4501. Expand z^4 and z^2. Use these to write the equation as a biquadratic in z.

4502. (a) Substitute α for both x_{n+1} and x_n.

(b) Show that the iteration is a GP with common ratio $r \in (0, 1)$.

4503. Separate the variables, multiplying up by $\sin^2 x$. Rewrite $\sin^2 x$ using a double-angle formula, and integrate. Substitute $(0, 0)$ in to find the constant of integration. Then rearrange to the required form.

4504. Complete the square to find the vertex of the parabola. The image is a monic parabola of the form $x = \mathrm{f}(y)$, with the same vertex.

4505. Take out $(x - 1)$ on top and bottom.

4506. The effect of raising $\cos x$ to the fifth power is to convert the x axis intercepts of $y = \cos x$ into points of inflection (quintuple roots). The minima and maxima remain the same, as do the positives and negatives.

4507. Draw a force diagram, and take moments around the axle. Be careful with the $\pm$ signs: depending on which angles you use, directions of moments may or may not be taken care of by $-$ve values of trig functions. Once you've got the moments equation, use compound-angle formulae to expand factors of the form $\sin(\theta \pm {}^{2\pi}/_{3})$. Rearrange to make $\tan\theta$ the subject.

4508. The graphs are reflections in the line $y = x$. Hence, the shortest distance between them must lie along a line of gradient $m = -1$.

4509. Simplify the integrand, writing it in the form

$$a + \frac{b}{x + c}.$$

Integrate and simplify with log rules.

4510. Let $y = \dfrac{2}{1 - \sqrt{x}}$.

Set up the usual first-principles limit. Multiply top and bottom of the fraction by the denominators of the inlaid fractions, and simplify. Then multiply top and bottom by the conjugate

$$C = \sqrt{x + h} + \sqrt{x}.$$

Calling it C simplifies the algebra a fair bit, which is quite heavy. Simplify the resulting difference of two squares, cancel a factor of h, and then take the limit.

4511. (a) Consider the case $x \geq 0$. The equation of the curve is then $(y - x)^2 + y^2 = 1$. Differentiate implicitly with respect to x. Set $\frac{dy}{dx}$ to zero and solve. Be careful to check any points: some phantom solutions will emerge.

(b) Repeat the process, this time differentiating with respect to y.

4512. (a) So, use Pythagoras to show that $r = kt$.

(b) One of the graphs is a spiral.

4513. Let $\frac{1}{2}\theta = \phi$. Write the RHS in terms of 2ϕ, and use double-angle identities to simplify. Choose the version of $\cos 2\phi$ that cancels the 1.

4514. The circles have radius 1, and are centred at $x = p$, $y = |p|$. This has Cartesian equation $y = |x|$. For a point to lie on at least one of the circles, it must lie within 1 unit distance of $y = |x|$.

4515. Substitute for z in the first equation, then solve using log rules.

4516. (a) Differentiate implicitly with respect to x, then use the second Pythagorean trig identity.

(b) Let $u = \arctan x$ and $v' = 1$.

4517. Consider a tangent line with equation $y = \mathrm{f}(x)$. This intersects the curve where

$$\begin{aligned} & x^3 - x = \mathrm{f}(x) \\ \iff & x^3 - x = \mathrm{f}(x) = 0. \end{aligned}$$

This is a cubic equation. You know that it has a repeated root at the point of tangency. This must be a double root or a triple root. Consider these case by case.

4518. Differentiate with respect to y. The negative of $\frac{dx}{dy}$ is the gradient of the normal. Equate this to $\frac{y}{x}$, which is the gradient of the line from $(0, 0)$ to (x, y). Solve the resulting equation in y, rejecting various phantom solutions.

4519. Call the colours RGBY. The orders of RGB are

RGB	BRG	GBR
RBG	BGR	GRB

Work out whether any of these are different from each other, and go from there.

4520. The DE is a quadratic in $\frac{dy}{dx}$.

4521. Put everything on the LHS, and factorise.

4522. Place the perpendicular sides of the triangle on the x and y axes, so the equation of the hypotenuse is $y = 3 - \frac{3}{4}x$. The centre of the circle is then (k, k), and it has equation

$$(x - k)^2 + (y - k)^2 = k^2.$$

You need exactly one simultaneous solution.

4523. (a) As x and y get large, the 1 becomes negligible compared to x^3 and y^3.

(b) Differentiate implicitly with respect to x. Set $\frac{dy}{dx} = 0$ and solve. Repeat with respect to y.

(c) Establish that the curve is asymptotic to $y = x$ in the positive and negative quadrants. Part (b) tells you how the curve departs from $y = x$.

4524. The quartic is invertible over two intervals whose union is $\mathbb{R}$. So, since a quartic must have a turning point, $\big(p, \mathrm{f}(p)\big)$ must be that turning point. Note also that, since f is monic, it is positive.

4525. The problem has two stages.

① The first stage is a basic pulley system with masses $6m$ and $4m$. This lasts for one second. Find the relative velocity and position of the monkeys after the first stage.

② The second stage is freefall for the victim and a basic pulley system with masses $6m$ and $2m$ for the thief. Find the relative acceleration for the second stage.

Then use a single *suvat*, relative to the thief, to find the time taken to close the gap.

4526. Write $x^x \equiv (e^{\ln x})^x \equiv e^{x \ln x}$.

4527. The problem is symmetrical, so you can work with $x \geq 0$. Find the x coordinates of the intersections, in terms of k where necessary. Then equate two definite integrals and solve for k.

4528. If it can be done, then $n \geq p + q$. There are p white, q black and $n - p - q$ empty squares. Take each of these as distinguishable, and you have n objects. Consider a list of the $n!$ orders of these objects, and describe the overcounting factors.

4529. Putting the RHS over a common denominator, the numerator must be linear.

4530. Assume the cubic is monic, wlog. So, its form is

$$y = (x - p)(x - q)(x - r).$$

Differentiate this twice by the product rule. You can use the three-way product rule, which is

$$(uvw)' = u'vw + uv'w + uvw'.$$

Set the second derivative to zero.

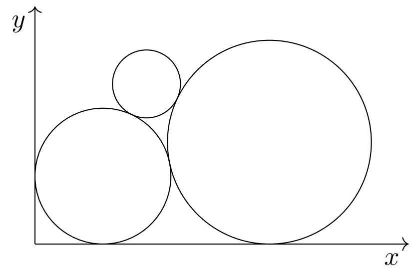
4531. Find the coordinates of the stationary point shown. Show that the value of given integral is greater than the area of a square.

4532. (a) It's another binomial distribution.

(b) Calculating via the individual distributions, there are three possibilities for (X, Y), namely $(0, 2)$, $(1, 1)$ and $(2, 0)$. Find the probability of each of these, by multiplying two binomial probabilities together.

4533. Consider the boundary case, in which the tension in the tow-bar is zero.

4534. The centres of the circles form a $(3, 4, 5)$ triangle. Rotate the picture so that the larger circles are tangential to the x axis.



Then consider a rectangle with sides parallel to the axes.

4535. The sum of the first n integers is $\frac{1}{2}n(n + 1)$. Use this in the LHS, simplifying to get the same result, with $n + 1$ in place of n, on the RHS.

4536. Consider the binomial expansion of $(1 + 1)^n$.

4537. (a) Find d and $\dot{d}$ at $t = 0$.

(b) Write the formula in the form

$$d = at + b + \frac{c}{10t + 25}.$$

4538. (a) Consider a small-angle approximation.

(b) The minimum occurs at the first SP for which $x > 0$. Set the derivative to zero, and you'll get a non-analytically solvable equation. Use the Newton-Raphson iteration to solve it.

4539. There are two cases to consider:

① $\mathrm{g}(x) > x$ on (α_k, α_{k+1}),

② $\mathrm{g}(x) < x$ on (α_k, α_{k+1}).

In each case, sketch $y = \mathrm{g}(x)$ and $y = x$ over the domain in question, and use a staircase diagram to visualise the behaviour of the iteration.

4540. There are four directions: horizontal/vertical and the two 45° diagonals. Consider these case by case. With the diagonals, there are subcases: diagonals of length $n-1$ or n.

4541. Multiply by $\sin x \cos x$ and square, noting that you may have introduced new solutions. Use identities to form a quadratic in $\sin 2x$.

4542. The radius is given by the greatest horizontal range. Take the result given, and set $y = 0$ and $x = r$. Rearrange to form a quadratic in $\tan\theta$. Maximum range is obtained when this has exactly one root, i.e. when $\Delta = 0$.

4543. Notate the moves L, R and N for left, right and no move. Fill in the rest of the following table:

Moves	Calculation	Probability
LLLL	$\frac{1}{4}^4$	$\frac{1}{256}$
RRRR	$\frac{1}{4}^4$	
LLRR	${}^4\text{C}_2 \times \frac{1}{4}^4$	
LRNN		
NNNN		

4544. The centres of the baubles are a distance $2r$ away from each other, so the three centres and the point from which the baubles are hung form a regular tetrahedron.

4545. Since the LHS is monic, and has a triple integer root, it must factorise as follows:

$$x^4 - 7x^3 + 18x^2 - 20x + 8 \equiv (x-a)^3(x-b).$$

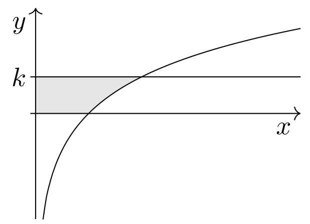
Equate coefficients of x^3 to show that $b \in \mathbb{Z}$. Then equate constant terms.

4546. (a) Consider an arbitrarily large value of y. Show that you can always find a point (x, y) on C_n with such a y value.

(b) Show that, if $|y| > 1$, then there are no points (x, y) on C_n.

4547. Write $\cos 3\theta$ as $\cos(2\theta + \theta)$.

4548. The region is



Set up an integral with respect to y.

4549. There are ${}^8\text{C}_4 = 70$ ways of selecting the edges. Count successful outcomes. There aren't many, so this can be done by argument or exhaustion.

4550. Rearrange the second equation, square and use the first Pythagorean identity. Form simultaneous equations in the variables $s = \sin x$ and $c = \cos y$. Solve these.

4551. Sketch the boundary equation. Then factorise the LHS of the inequality as a difference of two squares: such a product is positive if both factors have the same sign.

4552. This is a cubic in 3^x.

4553. The rope is smooth, so the transverse force cannot apply any friction to it. Hence, the force diagram must have a line of symmetry, i.e. the triangle of forces must be isosceles. Let $\phi = \frac{1}{2}\theta$. Write T in terms of $\cos\phi$ and P. Then use a double-angle formula to write $\cos\phi$ in terms of $\cos 2\phi$.

4554. A cubic graph has rotational symmetry around its point of inflection.

4555. Find the x intercept. Set up a definite integral and integrate by parts.

4556. (a) Use the chain rule in the form

$$\frac{dz}{dt} = \frac{dz}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

(b) Use the method of part (a) to find $\frac{dy}{dt}$. Then find $\frac{d}{dt}(x + y + z)$ and integrate.

4557. The factorial definition of ${}^n\text{C}_r$ is

$${}^n\text{C}_r = \frac{n!}{r!(n-r)!}.$$

The numerator has a factor of n. Consider whether the denominator can also have a factor of n.

4558. (a) The shaded region is a kite. Two of its sides have length 1. Find the length of the other two sides, using a tan compound-angle formula.

(b) The curve is symmetrical in $\theta = 180°$. Also, the squares do not overlap for $\theta \in [90°, 270°]$.

4559. (a) You can write this down without calculation.

(b) Sketch the curve, finding the coordinates of and classifying the stationary point.

4560. The RHS is the partial sum of a GP. Find the sum with the usual formula. Then set up the equation $x = \text{g}(x)$ for fixed points, and solve.

4561. Differentiate implicitly with respect to x or use circle geometry. Then use

$$y - y_1 = m(x - x_1).$$

4562. Factorise the RHS.

4563. (a) Exclude the root of the denominator.
   (b) Simplify $f^2(x)$ algebraically. Exclude from $\mathbb{R}$ anything for which $f^2(x)$ is undefined. Also, consider whether the value excluded in part (a) should also be excluded here.

4564. Just add up the probabilities of four outcomes:

$$\{\text{YYN}, \text{YNY}, \text{NYY}, \text{YYY}\}$$

4565. Consider each integral graphically.

——— Alternative Method ———

Evaluate with a double-angle formula.

4566. (a) A cube is rotationally symmetrical around its space diagonal.
   (b) Let x be the distance of each vertex of the hexagon from the nearest vertex of the cube.
   (c) Show that the hexagon has area

$$A = \tfrac{\sqrt{3}}{2}\big(1 + 2x - 2x^2\big).$$

   Then set $\dfrac{dA}{dx} = 0$.

4567. Sketch the graph of $y = g(x) = x^3 - x - 4$, together with the line $y = x$.

4568. This is a separable DE. Separate the variables and integrate (one side by parts).

4569. The substitution is $x + 1 = \tan\theta$.

4570. Differentiate to find the horizontal velocity when the field is switched off. Use a vertical *suvat* to find the time spent in projectile flight. Then set up a horizontal $x = x_0 + ut$. Write the resulting function in harmonic form to find its range.

4571. (a) The probability of a draw is $\frac{1}{3}$.
   (b) At least five rounds are required if the first four rounds produce at most one win.

4572. Start by factorising fully - the LHS can be expressed as the product of three linear factors. Each of these produces a straight line boundary equation. Then, consider the signs of the factors: for the product to be positive, you require an odd number of factors to be positive.

4573. (a) Solve $x = 0$ and $y = 0$.
   (b) Evaluate an integral of the form

$$\int_{t_1}^{t_2} y\frac{dx}{dt}\,dt.$$

4574. Set up $n(n+1)(n+2)(n+3)$. Expand this. Then add 1. You want the resulting expression to be a perfect square; it must be the square of a quadratic. Find this quadratic.

4575. (a) There is now a double root at $x = 0$.
   (b) Use the substitution $z = x^2$.
   (c) Use the substitution $z = x^3$.

4576. (a) Substitute into the DE.
   (b) Find the derivatives of $y = f(x)e^x$ using the product rule. Then substitute into the DE. Much should cancel.
   (c) Integrate $f''(x) = 0$ twice. Then substitute your result back into $y = f(x)e^x$.

4577. Place the first counter without loss of generality. Put it in the corner. This leaves an $(n-1)\times(n-1)$ square. The probability that the second counter is placed in this square is

$$\frac{(n-1)^2}{n^2-1}.$$

Continue in this fashion.

4578. The structure of the factorisation is

$$(ax + by + cz)(Ax + By + Cz).$$

Compare coefficients.

4579. You don't need (and it is quite tricky) to find all possible SPs. So, set up the equation for SPs and pick out the solutions you need.

4580. Use a combinatorial approach here. Consider the ${}^6\text{C}_4 = 15$ ways of choosing four numbers from the set $\{1, 2, 3, 4, 5, 6\}$.

4581. Use the parametric differentiation formula

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$$

to find the gradient at the origin. Then find the angle of inclination as $\arctan m$. Double this to find the relevant acute angle.

4582. Integrate by inspection, using $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

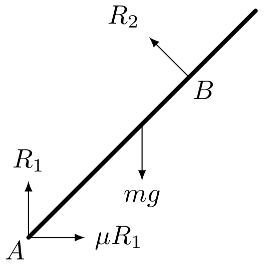
4583. Find the equation of the tangent, and show that $x_0 = \frac{n+2}{n}$. Sketch this graph to see the solution of $x_0 \geq 0$.

4584. (a) Differentiate both sides with respect to x.
(b) Integrate twice with respect to x.

4585. Equate two expressions for the gradients from the origin. Solve this equation, checking carefully the validity of the solutions.

4586. This is a quadratic in $|x|$. Solve it as $|x| = ...$ using the quadratic formula, and then consider the signs of the results.

4587. Consider limiting equilibrium, with friction at $F_{\max} = \mu R$. The force diagram for the ladder is



Set up three equations: horizontal, vertical and moments around A.

4588. Multiply numerator and denominator of the RHS by $\sin^2 x \cos^2 x$, then factorise the denominator.

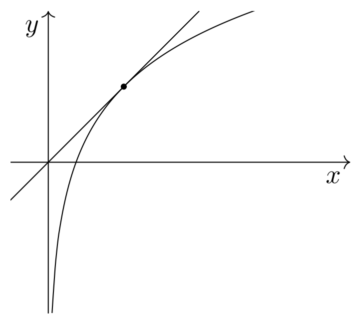
4589. (a) Consider parallel lines.
(b) Find, in terms of a, b, c, d, the coordinates of the intersection of the first two lines. Then sub these into the equation of the third line.

4590. This is a quadratic in e^x.

4591. If you simplify the boundary equation effectively, not much calculation is needed.

4592. Start with $x = \sin y$. Differentiate implicitly.

4593. The domain of f is $(0, \infty)$. The boundary case, at and beyond which x_1 is non-positive, occurs when the tangent to $y = \ln x + 1$ passes through O:



4594. Multiply out and simplify the integrand, using the identity $\cos^2 t \equiv \frac{1}{2}(\cos 2t + 1)$.

4595. (a) On the LHS, take out a common factor of $(n+1)^2$, then deal with the rest.
(b) In general, any partial sum S_{n+1} is produced by adding u_{k+1} to S_n.

4596. This is just a question of manipulating volumes etc. Firstly, work out the height of the cone in terms of r. Then the thing which requires care is the fact that the cone fills up from the bottom. Consider the empty space in the cone as a similar cone in its own right.

4597. Factorise the boundary equation. Use the fact that all of the roots are single roots to find the solution set. You don't need to sketch a graph, but it may well be helpful.

4598. Both excesses repeat after $[0, 2\pi]$, so you only need consider that domain. Consider the five cases

① $k \in (2, \infty)$,
② $k \in (0, 2)$,
③ $k = 0$,
④ $k \in (-2, 0)$,
⑤ $k \in (-\infty, -2)$.

4599. (a) Solve simultaneously. Use the fact that you know $x = -1$ is a root to factorise.
(b) Find the y coordinate at B first, simplifying fully.
(c) For the lines to form the setup for the angle in a semicircle theorem, you need to show that the normal passes through $(1, 0)$.

4600. (a) Substitute in the initial values of S and D.
(b) Find S and D as linear functions of t, equate them and solve.
(c) Find the values of S and D after one month, using your working in (b). Then repeat the same procedure, recalculating both rates at $t = 1$. Your equations, with numerical values for S_1, D_1, m_S and m_D, should be

$$S - S_1 = m_S(t - 1),$$
$$D - D_1 = m_D(t - 1).$$

Set $S = D$ and solve.

——— End of 46th Hundred ———